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Stable Nonequilibrium Probability Densities and Phase Transitions for Mean-Field Models in the Thermodynamic Limit

Luis L. Bonilla[1]

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A nonlinear Fokker–Planck equation is derived to describe the cooperative behavior of general stochastic systems interacting via mean-field couplings, in the limit of an infinite number of such systems. Disordered systems are also considered. In the weak-noise limit; a general result yields the possibility of having bifurcations from stationary solutions of the nonlinear Fokker–Planck equation into stable time-dependent solutions. The latter are interpreted as nonequilibrium probability distributions (states), and the bifurcations to them as nonequilibrium phase transitions. In the thermodynamic limit, results for three models are given for illustrative purposes. A model of self-synchronization of nonlinear oscillators presents a Hopf bifurcation to a time-periodic probability density, which can be analyzed for any value of the noise. The effects of disorder are illustrated by a simplified version of the Sompolinsky–Zippelius model of spin-glasses. Finally, results for the Fukuyama–Lee–Fisher model of charge-density waves are given. A singular perturbation analysis shows that the depinning transition is a bifurcation problem modified by the disorder noise due to impurities. Far from the bifurcation point, the CDW is either pinned or free, obeying (to leading order) the Grüner–Zawadowki–Chaikin equation. Near the bifurcation, the disorder noise drastically modifies the pattern, giving a quenched average of the CDW current which is constant. Critical exponents are found to depend on the noise, and they are larger than Fisher's values for the two probability distributions considered.

KEY WORDS: Nonlinear Fokker–Planck equation; mean-field model; nonequilibrium phase transitions; bifurcation and perturbation theory; charge-density waves; spin-glasses; oscillator self-synchronization.

[1] Departamento de Física Teórica, Universidad de Sevilla, 41080 Seville, Spain.

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1. INTRODUCTION

The appearance of order in a macroscopic system through fluctuations and interaction of subsystems is a much analyzed phenomenon in statistical physics and other sciences.[2-10] Equilibrium phase transitions are among the best-understood examples of these spontaneous ordering processes. In statistical physics, they are bifurcations between stable stationary probability distributions. These distributions may be associated with systems of infinitely many coupled stochastic equations.

For physical systems of finitely many stochastic equations, the H-theorem implies that any initial probability distribution evolves toward the unique stationary distribution: the equilibrium state.[1] Thus, only in the thermodynamic limit can we have equilibrium phase transitions.[2,3],[2] But we can have even more. It is possible to find stable time-dependent probability distributions and bifurcations to them from other distributions (stationary or not). In other words, in the thermodynamic limit, we can find nonequilibrium probability distributions and nonequilibrium phase transitions.[4,5]

In the literature the term "nonequilibrium phase transition" is sometimes used with a different meaning: a change in the number of maxima of the (unique) stationary solution of a standard Fokker–Planck equation (FPE) when some parameter varies (see, e.g., Ref. 6). The coordinates of these maxima are then identified with the order parameter of the phase transition. Although for each value of the control parameter there is one and only one stationary solution of the FPE (and thus a unique mean value), experimentally one would measure the values of the variable with maximum probability. This is *not* the meaning I intend in this paper: when the probability density bifurcates, there is more than one density that solves the nonlinear Fokker–Planck equation (see below). Multiplicity of solutions of standard Fokker–Planck equations (which are linear in the probability density) is impossible. The existence of stable time-dependent solutions of the standard Fokker–Planck equation is forbidden by the H-theorem.[1]

In this paper I analyze infinite systems of stochastic equations coupled via a mean-field interaction. I derive, quite generally, a nonlinear equation for the one-system probability density. Then I demonstrate the existence of nonequilibrium probability densities and phase transitions by solving this equation for some particular models.

[2] I will not distinguish in this paper between equilibrium and stationary states. The reason is that I do not want to use a baroque terminology such as "nonequilibrium phase transitions to stable time-dependent states," even though this is the exact meaning I have in mind.

Our starting point is a paper by Desai and Zwanzig.[2] They analyzed the following equations:

$$dx_j/dt = (1-x_j^2)\,x_j + F^{1/2}w_j(t) - J\left(x_j - N^{-1}\sum_{1\leqslant k\leqslant N} x_k\right), \qquad j=1,\ldots,N \tag{1.1}$$

Here the $w_j(t)$ are independent Gaussian white noises $[\langle w_j\rangle = 0$, $\langle w_j(t)\,w_k(t')\rangle = \delta_{jk}\,\delta(t-t')]$. With $J=0$, Eq. (1.1) describes the relaxation of a particle in a double-well potential, in contact with a thermal bath at temperature F. The last term in (1.1) can be viewed as an interaction between particles that creates a tendency for their coordinates x_j to relax toward the center of mass of the ensemble. Desai and Zwanzig derived the following nonlinear Fokker–Planck equation for the one-particle probability density:

$$\partial_t p(t;x) = \tfrac{1}{2}F\,\partial_x^2 p(t;x) - \partial_x\{[(1-x^2)\,x + J(\langle x(t)\rangle - x)]\,p(t;x)\} \tag{1.2a}$$

$$\langle x(t)\rangle = \int x p(t;x)\,dx \tag{1.2b}$$

$$\int p(t;x)\,dx = 1 \tag{1.2c}$$

They used the molecular chaos assumption $[p_2(t;x_1,x_2) = p(t;x_1)\,p(t;x_2)]$ to close a hierarchy of equations for all the multiparticle probability densities. They also gave another derivation that made it clear that (1.2) is asymptotically valid in the limit $N\to\infty$.[2] Later, Dawson proved this.[3] An analysis of (1.2) showed the existence of a pitchfork bifurcation for its stationary solutions, which corresponds to an equilibrium phase transition. Furthermore, the approach to equilibrium in the thermodynamic limit can be studied from (1.2).[2,3,19]

Here I derive nonlinear Fokker–Planck equations like (1.2) for more general mean-field models (having general nonlinearities, not just the double well, and with each x_j being an n-dimensional vector). Disordered systems with random coefficients in (1.1) also have a reduced equation for the one-particle density. I then study the nonlinear Fokker–Planck equation in the weak-noise limit. I discover that nonequilibrium densities may be expected when the system with $F=J=0$ has a time-dependent attractor. The precise form of these results (which are the central results of the paper) is given in Section 2. The rest of the paper is organized as follows. Section 3 contains results for models of self-synchronizing oscillators, spin-glasses, and charge-density waves. They are found by analyzing the corresponding nonlinear Fokker–Planck equation and con-

firm the existence of nonequilibrium densities and phase transitions. The purpose of this section is to illustrate the general results of Section 2. Thus, even though some of the models are interesting per se, I do not give a comprehensive analysis or list of results. I do this elsewhere.[4,5] The derivations of the results of Section 2 are in Section 4. Section 5 contains a rather sketchy derivation of the results of Section 3. Finally, I devote Section 6 to a discussion.

2. THE NONLINEAR FOKKER–PLANCK EQUATION. GENERAL RESULTS

Let us consider the following system of stochastic (Ito) equations:

$$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}) + F^{1/2}\mathbf{w}(t) \tag{2.1}$$

Here $\mathbf{x}(t) \equiv (x_1(t),..., x_n(t)), F > 0$, and $\mathbf{w}(t)$ is a Gaussian white noise [$\langle \mathbf{w}(t) \rangle = 0$, $\langle \mathbf{w}(t)\, \mathbf{w}(t') \rangle = \mathbf{1}\delta(t-t')$, $\mathbf{1}$ is the $n \times n$ identity matrix]. Equation (2.1) describes a dynamical system subject to a stochastic disturbance due to interaction with a bath at temperature F. Consider now a set of N systems identical to (2.1) interacting through a mean-field coupling. The resulting model is given by the equations

$$d\mathbf{x}_j/dt = \mathbf{f}(\mathbf{x}_j) + F^{1/2}\mathbf{w}_j(t) - J\left[\mathbf{x}_j - N^{-1} \sum_{1 \leqslant k \leqslant N} \mathbf{x}_k\right] \tag{2.2}$$

where $j = 1,..., N$ and $J > 0$, and noises corresponding to different subsystems are independent. The last term in (2.2) can be viewed as an interaction between subsystems of type (2.1) that creates a tendency for the subsystems to relax toward the center of mass of the ensemble.

Particular examples of the form (2.2) have been used in many contexts. Scalar subsystems with $f(x) = -x^3 + \alpha x$, the derivative of the ϕ^4 potential, have been used to study critical dynamics, chemical kinetics, muscle contraction, etc. (cf. Ref. 3 and references quoted therein). With additional time-independent noises, similar models have been proposed to describe the dynamics of spin-glasses[7] and the sliding of charge-density waves.[8]

The Fokker–Planck equation for the conditional probability density of the Markov process $[\mathbf{X}(t) \equiv (\mathbf{x}_1(t),..., \mathbf{x}_N(t))\}$ is

$$\begin{aligned} \partial_t P_N(t; \mathbf{X}, \mathbf{Y}) &= \tfrac{1}{2}F \sum_1^N \Delta_j P_N(t; \mathbf{X}, \mathbf{Y}) \\ &\quad - \sum_1^N \nabla_j \cdot \{[f(\mathbf{x}_j) + JN^{-1} \sum_k (\mathbf{x}_k - \mathbf{x}_j)]\, P_N(t; \mathbf{X}, \mathbf{Y})\} \end{aligned} \tag{2.3}$$

Here

$$\partial_t \equiv \partial/\partial t, \qquad \Delta_j \equiv \nabla_j \cdot \nabla_j \equiv \partial^2/\partial x_{1j}^2 + \cdots + \partial^2/\partial x_{nj}^2, \qquad \text{etc.}$$

The initial condition is

$$P_N(0; \mathbf{X}, \mathbf{Y}) = \delta(\mathbf{X} - \mathbf{Y}) \equiv \prod_1^N \delta(\mathbf{x}_j - \mathbf{y}_j)$$

The conditional probability is normalized so that $\int P_N(t; \mathbf{X}, \mathbf{Y})\, d\mathbf{X} = 1$. Let the initial data $\mathbf{Y}$ be distributed according to the molecular chaos assumption

$$\mu_N(\mathbf{Y}) = \prod_1^N \mu(\mathbf{y}_j) \tag{2.4}$$

At $t = 0$ the subsystems are uncorrelated, so the probability density of the ensemble is the product of the individual densities.

Result 1. The one-system density $p(t; \mathbf{x})$ defined below is, asymptotically in the limit $N \to \infty$, the solution of the following nonlinear Fokker–Planck equation:

$$\partial p(t; \mathbf{x})/\partial t = \tfrac{1}{2} F\, \Delta p(t; \mathbf{x}) - \nabla \cdot \{[f(\mathbf{x}) + J(\langle \mathbf{x}(t) \rangle - \mathbf{x})]\, p(t; \mathbf{x})\} \tag{2.5}$$

$$\langle \mathbf{x}(t) \rangle = \int \mathbf{x} p(t; \mathbf{x})\, d\mathbf{x}, \qquad \int p(t; \mathbf{x})\, d\mathbf{x} = 1 \tag{2.6}$$

$$p(0; \mathbf{x}) = \mu(\mathbf{x}) \tag{2.7}$$

$p(t; \mathbf{x})$ is defined by

$$p(t; \mathbf{x}_1) = \int P_N(t; \mathbf{X}, \mathbf{Y})\, \mu_N(\mathbf{Y})\, d\mathbf{Y}\, d\mathbf{x}_2 \cdots d\mathbf{x}_N \tag{2.8}$$

In the same limit, there is propagation of the molecular chaos, i.e., for any integer L,

$$P_L(t; \mathbf{x}_1, \ldots, \mathbf{x}_L) \equiv \lim_{N \to \infty} \int P_N(t; \mathbf{X}, \mathbf{Y})\, \mu_N(\mathbf{Y})\, d\mathbf{Y}\, d\mathbf{x}_{L+1} \cdots d\mathbf{x}_N$$

$$= p(t; \mathbf{x}_1) \cdots p(t; \mathbf{x}_L) \tag{2.9}$$

Suppose now that the function $\mathbf{f}(\mathbf{x})$ of Eq. (2.2) depends on random vector processes $\mathbf{h}_j$, with m components h_j^k ($k = 1, \ldots, m$; $j = 1, \ldots, N$) for each subsystem j. Then, the original stochastic equation is

$$d\mathbf{x}_j/dt = \mathbf{f}(\mathbf{x}_j; \mathbf{h}_j) + F^{1/2}\mathbf{w}_j(t) - J\left(\mathbf{x}_j - N^{-1} \sum_{1 \leqslant k \leqslant N} \mathbf{x}_k\right) \tag{2.10}$$

The noises $\mathbf{w}$ and $\mathbf{h}$ are independent. The $\mathbf{h}$'s in different points of space are uncorrelated, i.e., the probability distribution of the stochastic process $\{\mathbf{h}_j, 1 \leqslant j \leqslant N\}$ is

$$dP(\{\mathbf{h}_j, 1 \leqslant j \leqslant N\}) = \prod_1^N dP(\mathbf{h}_j) \tag{2.11}$$

Problems such as (2.10)–(2.11) are common in physics. They appear, for example, in models with random fields and models of charge-density waves. In these problems, one is often interested in quenched averages with respect to $\mathbf{h}$ of thermal averages, usually the mean or the two-point correlation of the $\mathbf{x}_j$,

$$\overline{\mathbf{x}_i(t)}, \qquad \overline{\mathbf{x}_i(t)\,\mathbf{x}_j(t')} \tag{2.12}$$

The overbar means average with respect to the thermal noise first and then with respect to $\mathbf{h}$. The $\mathbf{x}_j(t)$ are the solutions of (2.10) for given realizations of the two noises.

Result 2. Let us suppose the molecular chaos initial condition (2.4) for the N-system density. Now the μ's may depend on $\mathbf{h}$. As $N \to \infty$, the averages (2.12) are given by

$$\begin{aligned} \overline{\mathbf{x}_i(t)} &= \int dP(\mathbf{h}) \int d\mathbf{x}\, \mathbf{x} p(t, \mathbf{x}; \mathbf{h}) \\ \overline{\mathbf{x}_i(t)\,\mathbf{x}_j(t')} &= \delta_{ij} \int dP(\mathbf{h}) \int d\mathbf{x}\, d\mathbf{y}\, \mathbf{x}\mathbf{y} p(t, \mathbf{x}, \mathbf{y}; h) \end{aligned} \tag{2.13}$$

Here $p(t, \mathbf{x}; \mathbf{h})$ is a one-system density, which solves the equation and the initial and normalization conditions below:

$$\begin{aligned} \partial p(t; \mathbf{x}; \mathbf{h})/\partial t = \tfrac{1}{2} F\, \Delta p(t; \mathbf{x}; \mathbf{h}) - \nabla \cdot \Big\{ \Big[\mathbf{f}(\mathbf{x}; \mathbf{h}) \\ + J\left(\int \mathbf{y} p(t, \mathbf{y}; h)\, d\mathbf{y}\, dP(\mathbf{h}) - \mathbf{x} \right) \Big] p(t; \mathbf{x}; \mathbf{h}) \Big\} \end{aligned} \tag{2.14}$$

$$p(0, \mathbf{x}; \mathbf{h}) = \mu(\mathbf{x}; \mathbf{h}) \tag{2.15}$$

$$\int p(t, \mathbf{x}; \mathbf{h})\, d\mathbf{x}\, dP(\mathbf{h}) = 1 \tag{2.16}$$

$p(t, \mathbf{x}, \mathbf{y}; \mathbf{h})$ is a conditional one-system density that satisfies (2.14), (2.16), and the initial condition $p(0, \mathbf{x}, \mathbf{y}; \mathbf{h}) = \delta(\mathbf{x} - \mathbf{y})$, instead of (2.15).

Furthermore, if we denote by $\bar{P}_N(t; \mathbf{x}_1, ..., \mathbf{x}_N)$ the N-system density averaged with respect to all the $\mathbf{h}$'s, the following molecular chaos property holds:

$$\bar{P}_L(t; \mathbf{x}_1, ..., \mathbf{x}_L) \equiv \lim_{N \to \infty} \int \bar{P}_N(t; \mathbf{X})\, d\mathbf{x}_{L+1} \cdots d\mathbf{x}_N = \bar{p}(t; \mathbf{x}_1) \cdots \bar{p}(t; \mathbf{x}_L) \quad (2.17)$$

Here $\bar{p}(t; \mathbf{x}) = \int p(t, \mathbf{x}; \mathbf{h})\, dP(\mathbf{h})$.

Equation (2.14) corresponds to substituting $\int dP(h) \int d\mathbf{x}\, \mathbf{x} p(t, \mathbf{x}; h)$ for $N^{-1} \sum_{1 \leqslant k \leqslant N} \mathbf{x}_k(t)$ in (2.10), an intuitive result that has been conjectured on several occasions (see, for example, Ref. 8, where this conjecture plays an important role in the analysis). For Eq. (1.1) (which does not contain the extra noises $\mathbf{h}$), Dawson proved this central limit result.[3]

Result 3. Asymptotically as $F \to 0$, the solution of (2.14) is a functional of the mean $\overline{\mathbf{x}(t)}$, where $\overline{\mathbf{x}(t)}$ is the average with respect to $\mathbf{h}$ of the solution of the reduced equation

$$d\mathbf{x}(t; \mathbf{h})/dt = \mathbf{f}(\mathbf{x}(t; \mathbf{h}); \mathbf{h}) + J[\overline{\mathbf{x}(t)} - \mathbf{x}(t; \mathbf{h})], \qquad \mathbf{x}(0; \mathbf{h}) = \mathbf{y}(\mathbf{h}) \quad (2.18)$$

$$\overline{\mathbf{x}(t)} = \int \mathbf{x}(t; \mathbf{h})\, dP(\mathbf{h}) \quad (2.19)$$

The coordinate $\mathbf{y}(\mathbf{h})$ in (2.18) is that of the maximum of $\ln p(0, \mathbf{x}; \mathbf{h})$. The precise form of $p(t, \mathbf{x}; \mathbf{h})$ will not be written here. The two-time correlation (2.13) can be calculated by means of

$$\overline{\mathbf{x}(t)\, \mathbf{x}(t')} = \int \mathbf{x}(t - t'; \mathbf{y}; \mathbf{h})\, \mathbf{y}\, d\mathbf{y}\, dP(\mathbf{h}) \quad (2.20)$$

Here I have explicitly displayed the $\mathbf{y}$ dependence in the argument of the solution of (2.18)–(2.19). In the case of Eq. (2.5), we have $\overline{\mathbf{x}(t)} = \mathbf{x}(t; \mathbf{y}; \mathbf{h}) = \langle \mathbf{x}(t) \rangle$, and (2.18) becomes the simpler equation

$$d\langle \mathbf{x}(t) \rangle / dt = \mathbf{f}(\langle \mathbf{x}(t) \rangle) \quad (2.21)$$

From (2.21) the next result follows immediately:

Result 4. Let the deterministic equation $d\mathbf{x}/dt = \mathbf{f}(\mathbf{x})$ have an asymptotically stable solution $\mathbf{x} = \mathbf{A}(t)$. Then, asymptotically as $N \to \infty$ and $F \to 0$, the stochastic system (2.2) has a stable one-system probability density that is a time-dependent functional of $\mathbf{A}(t)$.

Result 4 is false for the standard stochastic equation (2.1). The stable solution of the Fokker–Planck equation associated with (2.1) is a stationary probability distribution that assigns equal probability to all

points $\mathbf{x}$ belonging to $\mathbf{A}(t)$, and probability zero to all other points. Thus, for the mean-field models we consider, nonequilibrium probability densities may exist in the thermodynamic limit only.

3. APPLICATIONS TO SOME SPECIFIC MODELS

In this section we use the nonlinear Fokker–Planck equation to analyze three models.

3.1. Self-Synchronization of Nonlinear Oscillators

Let us consider the following system of Langevin equations:

$$d\mathbf{x}_j(t)/dt = (\alpha - \mathbf{x}_j^2)\,\mathbf{x}_j + \mathbf{x}_j^\dagger + F^{1/2}\mathbf{w}_j(t) - J[\mathbf{x}_j(t) - N^{-1} \sum_{1 \leqslant k \leqslant N} \mathbf{x}_k(t)] \quad (3.1a)$$

$$\mathbf{x}_j \equiv (x_j,\, y_j), \qquad \mathbf{x}_j^\dagger \equiv (-y_j,\, x_j), \qquad \mathbf{x}_j \cdot \mathbf{x}_j^\dagger = 0 \quad (3.1b)$$

Here $j = 1,..., N$. For $J = F = 0$, the stable solution of (3.1a) is $\mathbf{x}_j = 0$ if $\alpha < 0$, and $\mathbf{x}_j = \alpha^{1/2}[\cos(t + \beta_j),\, \sin(t + \beta_j)]$ if $\alpha > 0$. Thus, for $\alpha > 0$, (3.1) is a collection of nonlinear oscillators subject to thermal noise fluctuations and coupled via a mean-field interaction. The nonlinear Fokker–Planck equation for the one-oscillator density is (2.5) with $\mathbf{f}(\mathbf{x}) = (\alpha - \mathbf{x}^2)\,\mathbf{x} + \mathbf{x}^\dagger$. For this model we have the following results[5]:

1. The stationary solution of the nonlinear Fokker–Planck equation is

$$p_s(\mathbf{x}) = Z^{-1} e^{-\phi(\mathbf{x})}, \qquad \phi(\mathbf{x}) = (J - \alpha + \tfrac{1}{2}\mathbf{x}^2)\,\mathbf{x}^2/F, \qquad Z = \int e^{-\phi(\mathbf{x})}\, d\mathbf{x} \quad (3.2)$$

2. The solution (3.2) is stable for $\alpha < JA + J$, where A obeys the equation:

$$A = \tfrac{1}{2}\theta^2 \left\{ 1 - 2\theta^{-1} \exp(-A^2/\theta^2) \left[\int_{-A/\theta}^{\infty} \exp(-r^2)\, dr \right]^{-1} \right\}, \qquad \theta = (2F)^{1/2}/J \quad (3.3)$$

3. At $\alpha = JA + J$, there is a supercritical Hopf bifurcation. For $\alpha > JA + J$, the asymptotically stable one-oscillator probability density is

$$p(t, \mathbf{x}) = Z^{-1} \exp\{ -\phi(\mathbf{x}) + [(\alpha/F - J^{-1})/(\tfrac{1}{2}\alpha - F/J)]^{1/2} \times (\alpha - J - JA)^{1/2}\, (y \cos t - x \sin t) + O(|\alpha - J - JA|) \} \quad (3.4)$$

The mean value of $\mathbf{x}$ is zero for $\alpha \leqslant JA + J$, and it is time-periodic for $\alpha > JA + J$:

$$\langle \mathbf{x}(t) \rangle = [(\alpha/F - J^{-1})/(\tfrac{1}{2}\alpha - F/J)]^{1/2} \times FJ^{-1}(\alpha - J - JA)^{1/2}(-\sin t, \cos t) + O(|\alpha - J - JA|)$$

Below the bifurcation point, the system of oscillators as a whole is in a quiescent state. Above it, the oscillators synchronize and a state of collective rhythmicity appears. Similarly, we can choose the noise F as the bifurcation parameter. Notice that the expansion in powers of $(\alpha - J - JA)^{1/2}$ is in the argument of the exponential in (3.4). This is not mere chance: if we try an expansion of $p(t, \mathbf{x})$ such as

$$p(t, \mathbf{x}) = Z^{-1} \exp\{-\phi(\mathbf{x})\} + (\alpha - J - JA)^{1/2} p_1(t, \mathbf{x}) + \cdots$$

(linear response theory) in the nonlinear Fokker–Planck equation, the periodic solution bifurcates vertically. This is true to all orders in $(\alpha - J - JA)^{1/2}$. In physical terms, trying to use nonlinear response theory to characterize the bifurcating periodic density fails to all orders.

The advantage of the system (3.1) is its computational simplicity, even though collective rhythmicity is expected for more involved nonlinearities $\mathbf{f}(\mathbf{x})$ such as van der Pol's nonlinearities.[9] In the more complicated example of the CDW that follows, only the $F \to 0$ limit will be considered.

3.2. A Simple Model of Spin-Glass Dynamics

The following equation is a (very) simplified version of Sompolinsky and Zippelius's model of spin-glass dynamics[7]:

$$dx_j/dt = (1 - x_j^2)\, x_j + F^{1/2} w_j(t) - (J + \theta)\left(x_j - N^{-1} \sum_{1 \leqslant k \leqslant N} x_k\right), \qquad j = 1, \ldots, N \tag{3.5}$$

Here θ is a zero-mean Gaussian noise of variance σ. All other coefficients and variables are as in (1.1). The original model[7] had a nearest neighbor interaction term with a random coupling J_{ij} instead of the last term in (3.5). The J_{ij} were time-independent Gaussian white noises. After averaging over the J_{ij}, the remaining sums over nearest neighbors were changed to mean-field forms (see Ref. 7 for details). Our version (3.5) is simpler, and it is included here for illustrative purposes only.

For each realization of the noise θ, and asymptotically as $N \to \infty$, we

derive the following nonlinear Fokker–Planck equation for the one-particle probability density:

$$\partial_t p(t; x; \theta) = \tfrac{1}{2} F \partial_x^2 p(t; x; \theta) \tag{3.6a}$$

$$- \partial_x \{ [(1 - x^2) x + (J + \theta)(\langle x(t) \rangle - x)] p(t; x; \theta) \} \tag{3.6b}$$

$$\langle x(t) \rangle = \int x p(t; x; \theta)\, dx$$

$$\int p(t; x; \theta)\, dx = 1 \tag{3.6c}$$

For each θ, the stable solution of (3.6) is that found by Desai and Zwanzig[2]: a stationary solution that experiences a pitchfork bifurcation. To calculate quenched averages, all we have to do is to average with respect to θ the corresponding moments of the density $p(t; x; \theta)$. For convenience, we choose $(J + \theta)$ as our bifurcation parameter. The graph $\langle x \rangle$ versus $(J + \theta)$ consists of a pitchfork bifurcation for each realization of θ. Then the bifurcation diagram, $\bar{x}$ versus J, consists of a pitchfork bifurcation smeared by the noise θ. The correlation $(\overline{x^2})^{1/2}$ measures the magnitude of this smearing. Far from the mean value of the bifurcation point, $(\overline{x^2})^{1/2} \sim \sigma^{1/2}$. Near the mean bifurcation point, $(\overline{x^2})^{1/2} \sim \sigma^{1/4}$, and fluctuations are much more important. In fact, they are of the same order as the mean $\bar{x}$. This is the picture of the "spin-glass transition" one achieves in this model. See Fig. 1 (Section 5).

3.3. Fisher's Model of Charge-Density Waves (CDW)[8]

This is a discretized mean-field version of the Fukuyama–Lee model of CDW.[10] It consists of the equations $(j = 1, \ldots, N)$

$$d\phi_j/dt = E - h_j \sin(\phi_j - \beta_j) - J\left(\phi_j - N^{-1} \sum_1^N \phi_k\right) + F^{1/2} w_j(t) \tag{3.7}$$

Here ϕ_j is the slowly varying phase of the CDW at the site j; J is the stiffness of the CDW; E is the applied electric field; and h_j and β_j represent random impurity potential and pinning angle, respectively. Their probability distributions factorize as in (2.11). The h_j and β_j are statistically independent. Then, asymptotically as $N \to \infty$, the one-phase probability density $p(t, \phi; h, \beta)$ satisfies (2.14)–(2.16), which now are

$$\partial_t p(t, \phi; h, \beta) = \tfrac{1}{2} F \partial_\phi^2 p(t, \phi; h, \beta)$$

$$- \partial_\phi \{ [E - h \sin(\phi - \beta) + J(\bar{\phi}(t) - \phi)] p(t, \phi; h, \beta) \} \tag{3.8}$$

$$\bar{\phi}(t) = (2\pi)^{-1} \int \phi p(t, \phi; h, \beta)\, d\phi\, d\beta\, dP(h) \tag{3.9}$$

$$p(0, \phi; h, \beta) = \mu(\phi; h, \beta) \tag{3.10}$$

$$(2\pi)^{-1} \int p(t, \phi; h, \beta)\, d\phi\, d\beta\, dP(h) = 1 \tag{3.11}$$

From these equations we have to find the quenched average of the CDW current, which is $\overline{d\phi/dt}$. This is the experimentally measurable current. Asymptotically as $F \to 0$, the reduced equations (2.18)–(2.19) for the extremal phase are (in dimensionless form)

$$\begin{aligned} d\theta(\tau; h, \beta)/d\tau &= G - \sin\theta + j[\bar{\theta}(t) - \theta + \bar{\beta} - \beta - H \sin\theta] \\ \theta(0; h, \beta) &= \sigma + \beta \end{aligned} \tag{3.12}$$

$$\begin{aligned} \theta(\tau; h, \beta) &\equiv \phi(t; h, \beta) - \beta \\ \tau = \bar{h}t, \qquad j = J/\bar{h}, \qquad G = E/\bar{h}, \qquad H &= (h - \bar{h})/J \end{aligned} \tag{3.13}$$

We have assumed that $\bar{h} \neq 0$. The overbar now means

$$\bar{f}(\tau; h, \beta) = (2\pi)^{-1} \int_{-\pi}^{\pi} d\beta \int dP(H)\, f(\tau; h, \beta) \tag{3.14}$$

With this definition $\bar{\beta} = 0 = \bar{H}$. We have analyzed (3.12)–(3.14) in the strong pinning limit $j \ll 1$, assuming that the correlation of H is at most $O(1)$. As argued by Fisher,[8] the CDW is pinned for small enough fields G only when $j < 1$. Therefore, to consider the strong pinning limit, $j \ll 1$, is not unreasonable. Furthermore, it makes life easier. In addition, we require that the correlation of the noise h_j be small in comparison with its mean value. Our problem is thus to ascertain the effect of the small j term of (3.12) in the single-phase equation. The solution is as follows[4]:

1. Far from the bifurcation point $G = 1$ (how far? $|G - 1| \gg j$), $\theta(\tau; h, \beta)$ differs little [how much? $O(j)$] from the solution of the unperturbed problem

$$d\theta_0(\tau)/d\tau = G - \sin\theta_0, \qquad \theta_0(0) = \sigma + \beta \tag{3.15}$$

(3.15) is the classical single-phase model of Grüner, Zawadowki, and Chaikin (see Ref. 8). Let us recall the behavior of the solution of this equation for $G > 0$. We consider angles θ_0 in the interval $[-\pi, \pi]$ only. It is sometimes useful to think of a cylindrical phase space $(\theta_0, d\theta_0/d\tau)$, where $\theta_0 \in [-\pi, \pi]$ and angles differing in integer multiples of 2π are identified.

For $G<1$, the solution monotonically tends to the constant $\theta_1=\sin^{-1} G$, $0<\theta_1\leqslant\pi/2$, unless the initial condition θ_0 coincides with the other constant solution of (3.15), $\theta_2=\pi-\theta_1$. In the cylindrical phase space, $(\theta_1, 0)$ is a stable node, while $(\theta_2, 0)$ is a saddle point. As G tends to 1 from below, saddle and node approach each other. At $G=1$, they coalesce, forming a saddle-node at $(\pi/2, 0)$. A loop formed by two of the separatrices of this saddle-node encircles the cylinder. For $G>1$, the constant solutions disappear and the separatrix loop becomes a stable time-periodic solution that is unique modulo a constant phase shift. As (3.15) can be solved by a quadrature, we have an exact description of this "creation of a limit cycle from the separatrix of a saddle-node." This global bifurcation is the depinning of the CDW in the bud. However, if $|G-1|\sim j$, (3.15) is not a good approximation to (3.12).

2. Near the bifurcation point $G=1$, a different asymptotic analysis has to be performed, considering now $|G-1|\sim j$. The result of this analysis to leading order is

$$\theta(\tau)\sim\pi/2+|2\Lambda|^{1/2}\tan(\tfrac{1}{2}|2\Lambda|^{1/2}\tau) - \sum_{m=0}^{\infty}\{2\cot^{-1}[\tau-|2\Lambda|^{-1/2}\pi(2m+1)] + 2/[\tau-|2\Lambda|^{-1/2}\pi(2m+1)]\} \tag{3.16}$$

$$\Lambda\equiv G-1-j(\beta+H) \tag{3.17}$$

Here we have chosen an initial condition according to which the probability that a phase ϕ takes the value $\beta+\pi/2$ is maximum. Otherwise we would have to shift the origin of time in (3.16).

3. For $|G-1|\gg j$, the quenched average of $d\theta/dt$ is equal to $d\theta_0/dt+O(j)$. Thus the excess current due to the CDW is zero if $G\ll 1$ and it is periodic if $G\gg 1$. For $|G-1|\sim j$ we must average the time derivative of (3.16)–(3.17) with respect to H and β to find out. Once this is done, we find

$$\overline{d\theta/dt}\sim(G-G_T)^{\zeta} \tag{3.18}$$

$$\begin{aligned}&\zeta=2 \quad \text{if} \quad dP(H)=\delta(H)\,dH\\ &\zeta=3 \quad \text{if} \quad dP(H)=\tfrac{1}{2}\eta(q-H)\,\eta(H+q)\,dH/q\end{aligned} \tag{3.19}$$

Here $\eta(x)$ is the Heaviside step function, equal to one for $x>0$ and equal to zero for $x<0$. The critical exponent ζ depends on the form of the distribution of the H's if J and the correlation of the h's are of the same order. If any of these parameters is much larger than the other (and both of them

are small compared to $\bar{h}$), $\zeta = 2$. If J and the correlation of the h's are of the same order, we have $\zeta = 3$ for H's distributed with equal probability on an interval $[-q, q]$. More results for this model will be given elsewhere.[4]

4. DERIVATION OF THE RESULTS OF SECTION 2

Results 1 and 2 can be derived by adapting a well-known method.[7] First, we insert in (2.8) the path integral representation of the solution of (2.3).[11] Manipulation of the resultant multiple path integral and use of the saddle-point method yields (2.5)–(2.7), asymptotically as $N \to \infty$. A similar derivation yields Result 2. Proofs of these results can be constructed by means of Dawson's methods.[3]

Derivation of Result 3. To solve (2.14)–(2.16) in the low-temperature limit, we use the WKB method, thereby inserting in (2.14) the following function:

$$p(t, \mathbf{x}; \mathbf{h}) = \exp\{-\Psi(t, \mathbf{x}; \mathbf{h})/F\}[Z_0(t, \mathbf{x}; \mathbf{h}) + FZ_1(t, \mathbf{x}; \mathbf{h}) + O(F^2)] \tag{4.1}$$

We find an eikonal equation for Ψ and linear transport equations for the Z's. The eikonal equation is

$$\partial_t \Psi + \tfrac{1}{2}(\nabla\Psi)^2 + [\mathbf{f}(\mathbf{x}; \mathbf{h}) + J(\bar{\mathbf{x}}(t) - \mathbf{x})] \cdot \nabla\Psi = 0 \tag{4.2}$$

$\bar{\mathbf{x}}(t)$ in (4.2) is given by (2.13) and (4.1). The solution of the initial value problem for Ψ is attained by the method of characteristics. It is

$$\Psi(t, \mathbf{x}; \mathbf{h}) = \Omega(\boldsymbol{\Sigma}(t, \mathbf{x}); \mathbf{h}) + \tfrac{1}{2}\int_0^t d\tau \left\{ \exp\left[\tau J \mathbf{1} - \int_0^\tau \nabla\mathbf{f}(\boldsymbol{\chi}(r, \mathbf{s}); \mathbf{h})\, dr \right] \nabla\Omega(\mathbf{s}; \mathbf{h}) \right\}^2 \Bigg|_{\mathbf{s} = \boldsymbol{\Sigma}(t, \mathbf{x})} \tag{4.3}$$

Here $\Omega(\mathbf{x}; \mathbf{h}) = \Psi(0, \mathbf{x}; \mathbf{h})$, and $\mathbf{x}(t, \mathbf{s})$ is the solution of the characteristic equation

$$d\boldsymbol{\chi}(t, \mathbf{s})/dt = \mathbf{f}(\boldsymbol{\chi}(t, \mathbf{s}); \mathbf{h}) + J[\bar{\mathbf{x}}(t) - \boldsymbol{\chi}(t, \mathbf{s})] + \exp\left[Jt\mathbf{1} - \int_0^t \nabla\mathbf{f}(\boldsymbol{\chi}(r, \mathbf{s}); \mathbf{h})\, dr \right] \nabla\Omega(\mathbf{s}; \mathbf{h}) \tag{4.4}$$

$$\boldsymbol{\chi}(0, \mathbf{s}) = \mathbf{s} \tag{4.5}$$

Once $\boldsymbol{\chi}(t, \mathbf{s})$ is known, $\boldsymbol{\Sigma}(t, \mathbf{x})$ is its inverse function (if it exists):

$$\boldsymbol{\chi}(t, \boldsymbol{\Sigma}(t, \mathbf{x})) = \mathbf{x}, \qquad \boldsymbol{\Sigma}(t, \mathbf{x}(t, \mathbf{s})) = \mathbf{s} \tag{4.6}$$

To evaluate $\bar{\mathbf{x}}(t)$, we use Laplace's method in (2.13). Asymptotically in the limit $F \to 0$, $\bar{\mathbf{x}}(t)$ is equal to the average with respect to $\mathbf{h}$ of that function $\mathbf{x} = \mathbf{x}_0(t; \mathbf{h})$ that minimizes $\Psi(t, \mathbf{x}; \mathbf{h})$. Thus $\bar{\mathbf{x}}(t)$ satisfies

$$\bar{\mathbf{x}}(t) = \int \mathbf{x}_0(t; \mathbf{h})\, dP(\mathbf{h}) \tag{4.7}$$

$\mathbf{x}_0(t; \mathbf{h})$ minimizes $\Psi(t, \mathbf{x}; \mathbf{h})$; therefore

$$\nabla \Psi(t, \mathbf{x}_0(t; \mathbf{h}); \mathbf{h}) = 0 \tag{4.8}$$

Assuming that $\boldsymbol{\Sigma}(t, \mathbf{x})$ exists, and considering (4.3), (4.8) is satisfied if $\nabla \Omega[\boldsymbol{\Sigma}(t, \mathbf{x}_0(t; \mathbf{h})); \mathbf{h}] = \mathbf{0}$. Suppose that the initial probability density has a single maximum at $\mathbf{s} = \mathbf{y}$. Then $\boldsymbol{\Sigma}(t, \mathbf{x}_0(t; \mathbf{h})) = \mathbf{y}$, which gives $\boldsymbol{\chi}(t, \mathbf{y}) = \mathbf{x}_0(t; \mathbf{h})$. By insertion in (4.4)–(4.5), we find

$$d\mathbf{x}_0(t; \mathbf{h})/dt = \mathbf{f}(\mathbf{x}_0(t; \mathbf{h}); \mathbf{h}) + J[\bar{\mathbf{x}}(t) - \mathbf{x}_0(t; \mathbf{h})], \qquad \mathbf{x}_0(0; \mathbf{h}) = \mathbf{y} \tag{4.9}$$

Equations (4.9) and (4.7) are (2.18) and (2.19), respectively. Equation (2.20) follows from Laplace's method applied to (2.13).

Derivation of Result 4. As mentioned in Section 2, this follows immediately from Result 3. I end this section by giving a local approximation to $p(t, \mathbf{x}; \mathbf{h})$, valid near $\mathbf{x} = \mathbf{x}_0(t; \mathbf{h})$. Assume that

$$\begin{aligned} p(t, \mathbf{x}; \mathbf{h}) \sim \exp\{ &- |\mathbf{x} - \mathbf{x}_0(t; \mathbf{h})|^2 / [2FW(t; \mathbf{h}; \boldsymbol{\gamma})] \\ &+ O(|\mathbf{x} - \mathbf{x}_0(t; \mathbf{h})|^3 / F)\} \end{aligned} \tag{4.10}$$

By inserting (4.10) in (2.14) and then ignoring $O[|\mathbf{x} - \mathbf{x}_0(t; \mathbf{h})|^3/F]$ terms, we find the following equation for $W(t; \mathbf{h}; \boldsymbol{\gamma})$:

$$\partial_t W(t; \mathbf{h}; \boldsymbol{\gamma}) + 2[J - \nabla \mathbf{f}(\mathbf{x}_0(t; \mathbf{h}); \mathbf{h}) : \boldsymbol{\gamma}^2]\, W(t; \mathbf{h}; \boldsymbol{\gamma}) = 1 \tag{4.11}$$

$\boldsymbol{\gamma}$ is an arbitrary unit vector. [From the derivation, $\boldsymbol{\gamma} = (\mathbf{x} - \mathbf{x}_0)/|\mathbf{x} - \mathbf{x}_0|$. But $\mathbf{x}$ is arbitrary, and so is $\boldsymbol{\gamma}$.] The $W(t; \mathbf{h}; \boldsymbol{\gamma})$ plays the role of a local correlation for (4.10). To solve (4.11), we need an initial or boundary condition for W. If $\mathbf{x}_0(t; \mathbf{h})$ is T-periodic in t, we impose that W also be T-periodic in t. For a more general attractor $\mathbf{x}_0(t; \mathbf{h})$, the time dependence of $W(t; \mathbf{h}; \boldsymbol{\gamma})$ is determined by that of $\mathbf{x}_0(t; \mathbf{h})$. It is expected for the Gaussian approximation (4.10) to break down near bifurcation points, since the operator $J\mathbf{1} - \nabla \mathbf{f}$ is not invertible there.

5. DERIVATION OF THE RESULTS IN SECTION 3

5.1. Self-Synchronization of Nonlinear Oscillators

Let us make the substitution $p(t, \mathbf{x}) = \exp[-\Phi(t, \mathbf{x})]$ in the nonlinear Fokker–Planck equation corresponding to (3.1). The resulting nonlinear equation is

$$-\partial_t \Phi(t, \mathbf{x}) = [\tfrac{1}{2} F \Delta \Phi(t, \mathbf{x}) + \nabla \cdot \mathbf{g}(\mathbf{x})] + \tfrac{1}{2} F [\nabla \Phi(t, \mathbf{x})]^2 + [\mathbf{g}(\mathbf{x}) + J \langle \mathbf{x}(t) \rangle] \cdot \nabla \Phi(t, \mathbf{x}) \tag{5.1}$$

$$\mathbf{g}(\mathbf{x}) = (\alpha - J - \mathbf{x}^2)\, \mathbf{x} + \mathbf{x}^\dagger \tag{5.2}$$

The stationary solution (3.2) solves $\frac{1}{2} F \Delta \Phi(t, \mathbf{x}) + \nabla \cdot \mathbf{g}(\mathbf{x}) = 0$. It also solves (5.1) with $\langle \mathbf{x} \rangle = 0$. It is a stationary state of type II, according to Jauslin's classification scheme[12] for the FPE. If $\langle \mathbf{x}(t) \rangle \equiv 0$, any initial condition of (5.1) relaxes oscillatorily to (3.2). In general, $\langle \mathbf{x}(t) \rangle \neq 0$, and (3.2) may be unstable. The linear stability analysis of (3.2) may be done as in the case of the model (1.1).[2,3] The result in dimensionless form is given by Eq. (3.3). Equation (3.4) is obtained by a standard Hopf bifurcation analysis of (5.1)–(5.2).

5.2. Spin-Glasses

As mentioned in Section 3, we can use Desai and Zwanzig's results[2] for (1.1). The presence of the noise θ shifts the bifurcation diagram back and forth (Fig. 2 of Ref. 2). We expect that the influence of θ will be most noticeable in a close neighborhood of the bifurcation point. To analyze this neighborhood, we take J near the bifurcation value in the absense of noise, J_0. Then we calculate the thermal mean value of x, $\langle x(\theta) \rangle$, for θ small. This is reasonable if the variance σ is small, which we assume to be the case. The calculation is similar to the derivation of Eq. (15) in Ref. 2. Near the bifurcation point, $\langle x(\theta) \rangle$ is a random variable given by

$$\langle x(\theta) \rangle = J_0^{-1} (3F/2)^{1/2} (J - J_0 + \theta)^{1/2} \eta(\theta) + O(|\theta|) \tag{5.3}$$

$\eta(\theta)$ is the Heaviside step function, equal to one if $\theta > 0$ and equal to zero if $\theta < 0$. The expected value with respect to θ of (5.3) gives us the quenched average $\bar{x}$. Recalling that θ is a zero-mean Gaussian variable, the probability distribution of $\langle x(\theta) \rangle$ is, approximately,

$$dP(\langle x(\theta) \rangle) = 4 J_0^2 (18 \pi \sigma F^2)^{-1/2} \langle x(\theta) \rangle \times \exp(-(2\sigma)^{-1} \{\tfrac{2}{3} F^{-1} [J_0 \langle x(\theta) \rangle]^2 + J_0 - J\}^2) \times \eta[\langle x(\theta) \rangle]\, d\langle x(\theta) \rangle \tag{5.4}$$

We have drawn Fig. 1 from (5.3) and (5.4). Near the bifurcation point J_0,

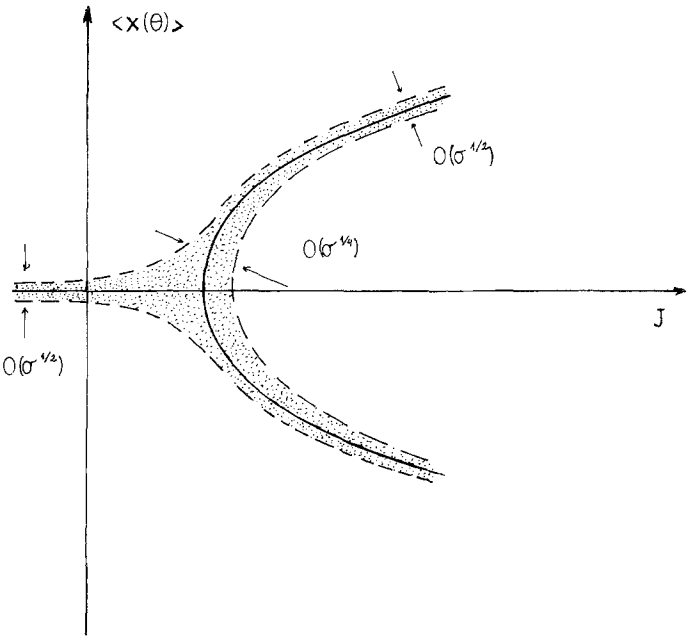

Fig. 1. A sketch of the thermal average $\langle x(\theta)\rangle$ versus J. The shaded region is the scatter band about the quenched average $\bar{x}$. Its width (proportional to the quenched correlation of $\langle x(\theta)\rangle$) is largest for J near the bifurcation point.

the quenched average $\bar{x}$ and the correlation of $\langle x(\theta)\rangle$ are of the same order: $O(\sigma^{1/4})$. This is much larger than the correlation far from the bifurcation point, which is $O(\sigma^{1/2})$. Far from the bifurcation point, a regular perturbation expansion says that $\langle x(\theta)\rangle$ is approximately linear in θ. Modifications of bifurcation diagrams due to small noises are studied more generally by Watson and Reiss.[13]

5.3. CDW

Far from $G=1$, a regular perturbation in powers of j suffices. The single-phase model (3.15) is then obtained as the leading order term. As $G\to 1$, secular terms (which are unbounded for $t\to\infty$) appear and therefore the perturbative scheme breaks down. Let us call the bifurcation region that interval of G where j and $G-1$ are of the same order. Let ε be a small parameter that characterizes the deviation of the phase θ from $\pi/2$ (the unperturbed phase θ_0 at $G=1$). Then the corrections to $\pi/2$ in the bifurcation region will be

$$\theta(\tau,\varepsilon)=\pi/2+\varepsilon\theta_1(\varepsilon^\gamma\tau)+O(\varepsilon^2)\equiv\pi/2+\varepsilon\theta_1(\zeta)+O(\varepsilon^2) \tag{5.5}$$

We have scaled the time anticipating a slower evolution in the bifurcation region. γ will be determined consistently in what follows. In the bifurcation region, the parameters j and $G-1$ are

$$j=\varepsilon^p j_p+O(\varepsilon^{p+1}), \qquad G=1+\varepsilon^p G_p+O(\varepsilon^{p+1}) \tag{5.6}$$

p and γ are now determined so that in the equation for θ_1, $d\theta_1/d\zeta$, j_p, G_p, and the leading nonlinear term are of the same order in ε (they balance each other). This distinguished limit corresponds to $\gamma=1$ and $p=2$. The equation for $\theta_1(\zeta)$ is

$$d\theta_1(\zeta)/d\zeta=G_2-j_2(\beta+H)+\tfrac{1}{2}\theta_1(\zeta)^2 \tag{5.7}$$

The solution of (5.7) yields the second term on the right side of (3.16). As the tangent blows up when its argument is an odd multiple of $\pi/2$, $\varepsilon\theta_1(\zeta)$ is not small compared to $\pi/2$ for neighborhoods of the corresponding times. Boundary layers then have to be inserted so that a bounded periodic $\theta(\tau;\varepsilon)$ is obtained. The third term in (3.16) is such a boundary layer correction. The remaining term causes (3.16) to be the leading-order uniform approximation to $\theta(\tau;\varepsilon)$. Similarly, an approximation to θ, uniform in G and τ, may be constructed from (3.16) and the solution of (3.15).[13]

The situation is now (to leading order) quite similar to that of our spin-glass model: we have a bifurcation diagram modified by a small, random imperfection. In this case the bifurcation is more complicated than a pitchfork bifurcation. It is the creation of a limit cycle from the separatrix loop of a saddle-node. Because of the time dependence, different definitions of the order parameter yield different critical exponents above the bifurcation point. For the CDW, the natural order parameter is the quenched average of the current $d\theta/d\tau$. By inserting (3.16) in the expression for $\overline{d\theta/d\tau}$, we find (3.18). The details will be given in Ref. 4. Note that the expression (3.18) is valid in the long-time limit only, after some transient terms have died out.

6. DISCUSSION

We have analyzed models of infinitely many identical subsystems interacting via a linear mean-field term, subject to thermal fluctuations and perhaps containing disorder. The simplicity of the mean-field coupling and the presence of the thermal noise make it possible to derive a nonlinear Fokker–Planck equation for the one-system probability density. For our derivation to work, it is crucial to use the molecular chaos initial condition (2.4). This situation is very common in derivations of macroscopic equations (cf. Boltzmann's), when only classes of initial conditions lead to evolutions described by the macroscopic equation.[14]

One problem not dwelt upon here is whether $N^{-1}\sum_{1\leqslant k\leqslant N}\mathbf{x}_k(t)$ tends

to $\langle \mathbf{x}(t) \rangle$ or to $\bar{\mathbf{x}}(t)$ in the thermodynamic limit. I conjecture that this central limit result is true for all those cases where the limits defining the path integrals exist. This is something to be shown for each model, as Dawson did for (1.1) in Ref. 3.

Our derivation can be easily extended to the case of multiplicative noise, where some function of $\mathbf{x}_j$ multiplies the thermal noise in (2.1) and successive equations. The only difference would have been a more complicated diffusivity in the resulting nonlinear FPE. Similarly, a tensorial-field coupling $\mathbf{J}(\mathbf{x}_j) \cdot (\mathbf{x}_k - \mathbf{x}_j)$ instead of $J(\mathbf{x}_k - \mathbf{x}_j)$ results in an obvious modification of the nonlinear FPE. Let the mean-field coupling be nonlinear: $N^{-1} \mathbf{J}(\mathbf{x}_j) \sum_1^N g(\mathbf{x}_k - \mathbf{x}_j)$ [$g(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are scalar and vector functions of $\mathbf{x}$, respectively]. The derivation of Section 4 still works for polynomial g's, or for g's that can be factorized so that the dependences in $\mathbf{x}_k$ and in $\mathbf{x}_j$ are separated. Examples are $g(\mathbf{x}) = \exp \mathbf{K} \cdot \mathbf{x}$, $\sin \mathbf{K} \cdot \mathbf{x}$,.... The consistency conditions are now more complicated than in the linear case: in general, higher moments, response functions, and correlations of the $\mathbf{x}$'s enter the nonlinear FPE. A similar situation was found by Sompolinsky and Zippelius[7] in their analysis of the dynamics of spin-glasses.

The nonlinear Fokker–Planck equation was already derived by Desai and Zwanzig in 1978[2] for the model (1.1). The present derivation, valid for more general models, led to the prediction of the existence of stable time-dependent probability densities (nonequilibrium densities) and bifurcations to them (nonequilibrium phase transitions). These phenomena appear in the thermodynamic limit only, as is also the case with equilibrium phase transitions.

Kuramoto[15] postulated a nonequilibrium phase transition in relation to the appearance of collective rhythmicity in systems of nonlinear oscillators (without disorder). His argument rested upon considering the mean-field interaction and the thermal noise as small corrections to the self-interaction of the oscillators. The effects of the thermodynamic limit were conjectured. As it turns out, the method of Section 4 yields an equation for the one-oscillator density that is different from Kuramoto's. There is still a transition to a synchronized state, although the details are quite different from those in Ref. 15.[5]

Derivation of the nonlinear Fokker–Planck equation reduces an N ($N \to \infty$)-body linear problem to a one-body nonlinear problem. In this process different noises are treated differently: (1) thermal or external noises, which are time- and space-dependent, contribute to the diffusivity of the nonlinear Fokker–Planck equation; (2) parametric noises (disorder), which are not time-dependent, become deterministic parameters. Their probability distributions enter the definition of the quenched average in the nonlinear Fokker–Planck equation.

Frequently, the nonlinear Fokker–Planck equation can be analyzed directly. If we find a stationary density, a linear stability analysis yields the bifurcation points of this density. Then we approximate it near these points by using the distance to the bifurcation point as a small parameter. Far from the bifurcation points, the Gaussian approximation works.[2,5] This approach works for problems with or without disorder noise. Sometimes all we can do is to study the weak-noise (i.e., the zero-temperature) limit. Then, to leading order in F (the temperature), the behavior of the density is given by the solution of a system of ordinary differential equations subject to some consistency conditions. This is the case for Fisher's model of CDW, which yields Eqs. (3.12)–(3.14). In problems without disorder noise, these reduced differential equations are the same as those for a deterministic uncoupled subsystem, Eq. (2.1) with $F=0$. In problems with disorder, the reduced equations are more complicated: (2.18)–(2.19).

If the disorder is weak, we can treat it as a perturbation (cf. Section 3). The effect of the disorder will be small for all values of the parameters except near bifurcation points, where it has a considerable importance. The idea of treating the disorder as a small disturbance also works if we have a more detailed description of the density, valid for any value of the temperature F.[5]

The CDW model of Section 3 was analyzed by Fisher.[8] His results were based on assuming that the quenched average of the CDW phase is linear in the time above the depinning field. Then he expanded the solution of (3.12)–(3.14) in powers of $v \equiv \overline{d\phi/dt}$. He explicitly excluded the possibility that (3.12)–(3.14) had time-periodic solutions. Supposedly, such solutions would disappear in the thermodynamic limit. The existence or nonexistence of periodic of periodic CDW current in the thermodynamic limit has originated some controversy in the literature.[8,16,17] The present results show that it is possible [and indeed the case for Eq. (3.7) in the strong pinning regime] for the oscillatory current to have persisted after the thermodynamic limit. For a different model, Sneddon[16] showed that no oscillations persist in the infinite volume limit. A calculation like that of Ref. 16 would yield the wrong result when carried out for (3.12)–(3.14) (with $N \to \infty$ substituting the infinite volume limit). In fact, Sneddon's calculation was based on the assumption that the pinning potential h was small: an expansion of the CDW phase in powers of h was followed by taking the infinite volume limit term by term. For (3.12)–(3.14) no oscillations survive this process. This is a spurious result here: Sneddon's is, in this case, an expansion about a singular limit of Eqs. (3.12)–(3.14). The leading order term in an h-expansion of the CDW phase corresponds to the limit $E \to \infty$. Why is this limit singular? For strong applied fields, (3.15) approximates well (3.12). When $1/E=0$, there are no oscillatory solutions

of (3.15), whereas for $1/E \neq 0$ (no matter how small $1/E$ is), the solution of (3.15) is time periodic.

Another prediction from these results is that the critical exponents are not universal (except if h is deterministic) and that $\zeta \geqslant 2$ (Sections 3 and 5). This latter value seems closer than Fisher's $\zeta = 3/2$ to the experimental ones in Ref. 18. More comments and results will be published elsewhere.[4]

It would be very interesting to find out how to describe non-equilibrium densities and phase transitions without the mean-field approximation. Unfortunately, this is quite hard. No technique of the same power as bifurcation theory exists for short-range interactions. The renormalization-group approach (a natural candidate) seems to be limited to equilibrium phase transitions and critical dynamics so far.

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